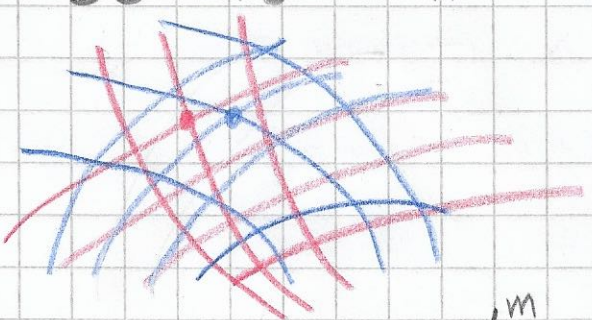
SUSSKIND GR 2

CHANGING COORDINATES

$$x^m \longleftrightarrow y^m$$

Scalars $S'(y) = S(x)$



$$y^m = y^m(x)$$

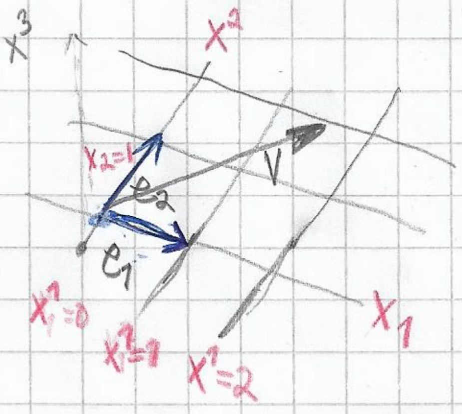
$$x^n = x^n(y)$$

COORD. TRANSF.:N
- ASSUME CONTINUOUS
DIFFERENTIABLE

V^m

V_m

TWO SCALAR FIELDS



$$\vec{e}_1 \ \vec{e}_2 \Rightarrow e_i$$

TAKE AN ARB. VECTOR

$$\vec{V} = V^1\vec{e}_1 + V^2\vec{e}_2 \left(+V^3\vec{e}_3\right) \ldots = V^m\vec{e}_m$$

= "$\vec{V}$ AS $\vec{e}$"

$$\vec{V}_n = V \cdot \vec{e}_n = V^m \underbrace{\vec{e}_m \cdot \vec{e}_n}_{=\text{metric tensor}}$$

$$V \cdot V = V^m e_m \cdot V^n e_n = V^m V^n (\vec{e}_m \cdot \vec{e}_n)$$

$$= V^m V^n g_{mn}$$

COVARIANT
CONTRAVARIANT
INDICES

i.e. $dx^m dx^n g_{mn}$

30min.

"TENSOR IS SOMETHING THAT TRANSFORMS LIKE A TENSOR UNDER COORDINATE TRANSFORMATIONS"

prime = y
unprimed = x

$$(V')^m = \frac{\partial y^m}{\partial x^n} V^n$$

$$dy^m = \frac{\partial y^m}{dx_n} dx_n$$

$$(W')_m = \frac{\partial x^n}{\partial y^m} W_n$$

$$\frac{\partial S}{\partial y^m}$$ GRADIENT OF S

$$= \frac{\partial x^n}{\partial y^m} \frac{\partial S}{\partial x_n}$$

NOTICE HOW THE NOTATION CARRIES YOU ALONG!

SEE THE SYMMETRY!

YOU GET A FEEL FOR THESE IN A WHILE!

$$(W')^m_n = \frac{\partial y^m}{\partial x^p} \frac{\partial x^q}{\partial y^n} W^p_q$$

$$(W')_{mn} = \frac{\partial x^p}{\partial y^m} \frac{\partial x^q}{\partial y^n} W_{pq}$$

SUSSKIND GR 2 page 2

"TENSOR IS AN OBJECT WHICH IS CHARACTERISED BY ITS TRANSFORMATION PROPERTIES"

- IF A SCALAR IS ZERO IN ONE FRAME IT'S ZERO IN ALL FRAMES
- $\vec{V} = 0 \Rightarrow (V^1, V^2, \ldots) = (0, 0, \ldots)$

"TENSOR EQUATIONS ARE TRUE IN ANY COORDINATE SYSTEM"

- THERE ARE OTHER OBJECTS – WE'LL SEE!

$$W^p_q = T^p_q$$

$W^p_q = T^{pq}$
YOU CAN WRITE, BUT ITS NOT A TENSOR! FOLLOW THE CONVENTION! INDICES BALANCE!

OPERATIONS ON TENSORS

15 min

MAKE NEW TENSORS! SKIP TRIVIAL: SCALING $\rightarrow SV^m = T^m$ NO RANK CHANGE

1. ADDITION
2. MULTIPLICATION
3. CONTRACTION
4. COVARIANT DIFFERENTIATION (WRT POSITION)

IND. MATCH!

$$\boxed{T^{m\ldots}_{\ldots p} \pm S^{m\ldots}_{\ldots p} = (T \pm S)^{m\ldots}_{\ldots p}} \quad (1.)$$

$$V^m \otimes W_n = T^m_n, \quad V^m \otimes W^n = T^{mn}$$

"MAKING TENSOR OF HIGHER RANK"

$$(2.) \quad V^m W^n = W^n V^m = T^{mn} \neq V^n W^m$$

LEMMA:

$$\sum_m \frac{\partial x^b}{\partial y^m} \frac{\partial y^m}{\partial x^a} = \frac{\partial x^b}{\partial x^a} = \delta^b_a$$

(3.)

$$V^m W_n = T^m_n$$

$$(V^m W_n)' = \frac{\partial y^m}{\partial x^a} \frac{\partial x^b}{\partial y^n} (V^a W_b)$$

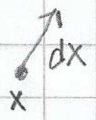
$$(V^m W_m)' = \frac{\partial y^m}{\partial x^a} \frac{\partial x^b}{\partial y^m} V^a W^b = V^a W_a$$

$\leftarrow$ SCALAR

SUSSKIND GR. 2 page 3.

4.

dx^m DIFFERENTIAL DISPLACEMENT



$ds^2 = g_{mn}(x)\, dx^m dx^n$ 10 COMPONENTS : $4 \times 4 = 16$ SYMMETRY $\Rightarrow 10$

LET'S PROVE g_{mn} IS A TENSOR

	X_1	X_2	X_3	X_4
X_1				
X_2				
X_3				
X_4				

$g_{mn}(x)\, dx^m dx^n$

$$= \boxed{g'_{pq}(y)}\, dy^p dy^q$$ ⟸ 2 ERI KOORDINAATISTOA

$$dx^m = \frac{\partial x^m}{\partial y^p} dy^p \qquad dx^n = \frac{\partial x^n}{\partial y^q} dy^q$$

$$= \boxed{g_{mn}(x) \frac{\partial x^m}{\partial y^p} \frac{\partial x^n}{\partial y^q}}\, dy^p dy^q$$

$$g_{mn} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

"WHAT DO YOU KNOW ABOUT MATRICES THAT ARE SYMMETRIC & HAVE NO ZERO EIGENVALUES?

- THEY ARE
- THEY'RE INVERTIBLE

g^{mn}

$g_{1n} g^{n1} = 1$

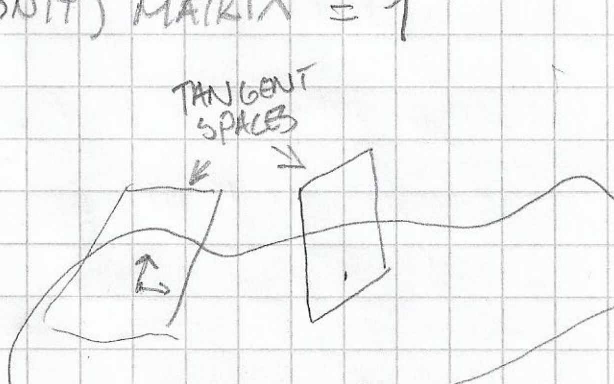
$g_{1n} g^{n2} = 0$

$$g_{mn} \boxed{g^{np}} = \delta^p_m \quad = \text{IDENTITY (UNIT) MATRIX} = 1$$

INVERSE

1h30min

GETTING THE NOTATION, FOLLOWING THE NOTATION!



"TENSORS LIVE IN TANGENT SPACES

WRITE OUT 2-by-2

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11} g^{11} + g_{12} g^{21} & g_{21} g^{12} + g_{21} g^{21} \\ g_{21} g^{11} + g_{22} g^{21} & g_{21} g^{12} + g_{22} g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$